REPRESENTATIONS OF THE UNIVERSAL COVERING GROUP OF THE GROUP $SU(1,q)$ [1]

V. F. Molchanov, Yu. A. Sharshov

G. R. Derzhavin Tambov State University, Russia
University of Leiden, Leiden, The Netherlands

In this paper we study a family of representations of the group $\widetilde{SU}(1,q)$, the universal covering group of the group $SU(1,q)$, induced by characters of a maximal parabolic subgroup. These representations can be realized on spaces of homogeneous functions on a cone in $\mathbb{C}^{1+q}$, so we call them representations associated with a cone. They are labelled by two complex parameters σ, τ.

We determine when these representations $T_{\sigma,\tau}$ are irreducible and describe completely the structure of invariant subspaces and irreducible subfactors (composition series) in the reducible case. We find out all intertwining operators and write down them both in a "matrix" and an integral form. We determine when an invariant sesqui-linear form exists for the pair $T_{\sigma,\tau}, T_{\sigma_1,\tau_1}$ (or their subfactors) and find out all unitarizable representations $T_{\sigma,\tau}$ or their subfactors. There are the following series of unitarizable representations: continuous series (2 real parameters), complementary series (2 real parameters), several "thin" series (which are "long" and "short") (1 real parameter), "discrete" series (discrete infinite set), "exceptional" series (discrete infinite set, the set of weights has lower "dimension").

The detailed description of these representations is necessary for requirements of harmonic analysis and quantization on the complex hyperbolic spaces $SU(1,q)/U(1,q-1)$.

For the groups $SU(p,q)$, the study of representations of class 1 with respect to $H = S(U(p,q-1)\times U(1)) \cong U(p,q-1)$ was undertaken in [5]. A description of representations of $SU(p,q), p>1$, of the class ω with respect to the same H was given in [8], partially it was done in [3]. The case of another H was considered in [7] for $\widetilde{SU}(n,n)$, $H = SL(n,\mathbb{C})\cdot\mathbb{R}^*$. See also [7] for references.

Remark that for the pair $SU(p,q)/U(p,q-1)$ the case $p=1$ has some peculiarities in comparing with the case $p>1$: another weight lattice, 2 complex parameters against 1 complex and 1 integer.

1 The group $SU(1,q)$ and its universal covering group

The group $G = SU(1,q)$ consists of matrices $g \in SL(n,\mathbb{C})$, $n = 1+q$, preserving the Hermitian form in $\mathbb{C}^n$:

$$[x,y] = -x_1\overline{y}_1 + x_2\overline{y}_2 + \cdots + x_n\overline{y}_n.$$

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We shall assume $q > 1$ (i.e. $n > 2$). We shall consider that G acts on $\mathbb{C}^n$ from the right and so we shall write vectors in the row form. Thus we have

$$\overline{g}' I g = I,$$

where $I = \operatorname{diag}\{-1, 1, \ldots, 1\}$, the prime denotes matrix transposition. Let us write $g \in G$ in the block form with respect to the partition $n = 1 + q$:

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{1.1}$$

so that α is a number.

The Lie algebra $\mathfrak{g}$ of the group G consists of complex $n \times n$ trace zero matrices X satisfying the condition

$$\overline{X}' I + IX = 0.$$

Let Herm (r) denote the space of Hermitian $r \times r$ matrices. A matrix X in $\mathfrak{g}$ has the following block form (as (1.1)):

$$X = \begin{pmatrix} i\xi & \eta \\ \overline{\eta}' & i\zeta \end{pmatrix}, \tag{1.2}$$

where $\xi \in \mathbb{R}$, $\zeta \in$ Herm (q), $\xi + \operatorname{tr} \zeta = 0$, η is a row-vector in $\mathbb{C}^q$.

Let us consider the two following commuting automorphisms θ and σ of G (for the corresponding automorphisms of $\mathfrak{g}$ we preserve notation θ and σ):

$$\theta(g) = IgI, \quad \sigma(g) = JgJ,$$

where $J = \operatorname{diag}\{1, \ldots, 1, -1\}$. Notice that both are inner, indeed, for example, $\theta(g) = TgT^{-1}$, where $T = \operatorname{diag}\{\lambda, -\lambda, \ldots, -\lambda\}$, $\lambda = \exp(i\pi q/n)$. Let K and H denote the fixed point subgroups of θ and σ, and $\mathfrak{k}$ and $\mathfrak{h}$ their Lie algebras respectively. The coset spaces G/K and G/H are semisimple symmetric spaces belonging to the class of complex hyperbolic spaces. See, for example, [6] in connection with that. The Lie algebra $\mathfrak{g}$ decomposes into the direct sums of $+1, -1$-eigenspaces of θ and σ respectively:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{h} + \mathfrak{q}.$$

The subgroup K consists of block diagonal matrices

$$\begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} \tag{1.3}$$

where $|\varphi| = 1$ and $\varphi \cdot \det \psi = 1$. It is isomorphic to $U(q)$ and is a maximal compact subgroup of G. The Lie algebra $\mathfrak{k}$ consists of block diagonal matrices

$$X = \begin{pmatrix} iu & 0 \\ 0 & i\zeta \end{pmatrix}, \tag{1.4}$$

where $u \in \mathbb{R}$, $\zeta \in$ Herm (q), $u + \operatorname{tr} \zeta = 0$. It has one-dimensional centre $Z(\mathfrak{k})$ spanned by

$$X_0 = \frac{i}{n} \begin{pmatrix} -q & 0 \\ 0 & E \end{pmatrix}. \tag{1.5}$$

Thus, G/K is a Hermitian symmetric space. The semisimple part $\mathfrak{k}^{(s)} = [\mathfrak{k}, \mathfrak{k}]$ consists of matrices (1.4) with $u = 0$, so that $\operatorname{tr} \zeta = 0$. The algebra $\mathfrak{k}$ decomposes into the direct sum

$$\mathfrak{k} = Z(\mathfrak{k}) + \mathfrak{k}^{(s)}. \tag{1.6}$$

Let $K^{(s)}$ be the subgroup of K consisting of matrices (1.3) with $\varphi = 1$ so that $\det \psi = 1$ and $K^{(s)}$ is isomorphic to $SU(q)$. Its Lie algebra is $\mathfrak{k}^{(s)}$.

The subspace $\mathfrak{p}$ consists of matrices

$$X = \begin{pmatrix} 0 & \eta \\ \overline{\eta}' & 0 \end{pmatrix},$$

where η is a row in $\mathbb{C}^q$.

Similarly the subgroup H consists of block diagonal matrices which are split into blocks according to the partition $n = q + 1$. Its Lie algebra $\mathfrak{h}$ has one-dimensional centre with a basis $\operatorname{diag}\{i, \ldots, i, -qi\}$. So that G/H is a semi-Kählerian symmetric space (in terminology of Berger [1], it means that it has G-invariant complex structure and G-invariant pseudo-Hermitian metric). The rank of G/H is equal to 1. It means that Cartan subspaces in $\mathfrak{q}$ have dimension 1. As this one let us take the subspace $\mathfrak{a}$ of $\mathfrak{q}$ with the basis:

$$A_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{1.7}$$

Here and further a matrix 3×3 denotes a matrix $n \times n$ written in the block form according to the partition $n = 1 + (n-2) + 1$.

The Lie algebra $\mathfrak{g}$ decomposes into the direct sum of root spaces of the pair $(\mathfrak{g}, \mathfrak{a})$:

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{g}_0 + \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}.$$

The subspace $\mathfrak{g}_0$ is the direct sum $\mathfrak{m} + \mathfrak{a}$ where the subalgebra $\mathfrak{m}$ consists of matrices

$$\begin{pmatrix} iu & 0 & 0 \\ 0 & iv & 0 \\ 0 & 0 & iu \end{pmatrix}, \tag{1.8}$$

where $u \in \mathbb{R}$, $v \in$ Herm $(n-2)$, $2u + \operatorname{tr} v = 0$. We see that $\mathfrak{m}$ lies in $\mathfrak{k}$ and is the direct sum of two commuting subalgebras:

$$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1,$$

where $\mathfrak{m}_0$ consists of matrices (1.8) with $u = 0$ (so that $\mathfrak{m}_0 \subset \mathfrak{k}^{(s)}$) and $\mathfrak{m}_1$ is the one-dimensional centre $\mathbb{R}Y_0$ of $\mathfrak{m}$ where

$$Y_0 = \begin{pmatrix} i & 0 & 0 \\ 0 & (-2i/(n-2))E & 0 \\ 0 & 0 & i \end{pmatrix},$$

The nilpotent subalgebra $\mathfrak{z} = \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$ corresponding to negative roots consists of matrices:

$$\begin{pmatrix} it & \xi & it \\ \overline{\xi}' & 0 & \overline{\xi}' \\ -it & -\xi & -it \end{pmatrix},$$

where $t \in \mathbb{R}$, ξ is a row in $\mathbb{C}^{n-2}$.

Let Y denote the cone $[y,y]=0$, $y \neq 0$. The group G acts on Y by translations: $y \mapsto yg$. Let S be the submanifold of Y defined by the equation $|y_1|=1$. It is the direct product of the circle and the sphere: $S = S^1 \times S^{2q-1}$. The group G acts on S as follows:

$$s \mapsto \widetilde{s} = s \cdot g = \frac{sg}{|(sg)_1|}. \tag{1.9}$$

In particular, the group K acts by translations: $s \mapsto s \cdot k = sk$.

Let L_X be the corresponding action of $\mathfrak{g}$:

$$(L_X\, f)\,(s) = \left.\frac{d}{dt}\right|_{t=0} f(s \cdot \exp tX), \quad X \in \mathfrak{g}. \tag{1.10}$$

In particular, we have

$$L_X(sg)_1 = (sX)_1,$$

$$L_X|(sg)_1| = \mathrm{Re}\,\langle sX, s\rangle, \tag{1.11}$$

where $\langle x,y\rangle$ denotes the following Hermitian form on $\mathbb{C}^n$:

$$\langle x,y\rangle = x_1\overline{y}_1. \tag{1.12}$$

Let us take as a basic point of S the point

$$s^0 = (1,0,\ldots,0,1).$$

The stabilizer of s^0 in $\mathfrak{g}$ for the action (1.9) is $\mathfrak{m}_0 + \mathfrak{a} + \mathfrak{z}$.

Let A, M, M_0, M_1, Z denote the analytic subgroups of G with the Lie algebras $\mathfrak{a}, \mathfrak{m}, \mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{z}$ respectively. For the action (1.9) the stabilizer of the point s^0 in G is M_0AZ and in K is M_0, so that $S = G/M_0AZ$ and

$$S = K/M_0.$$

Since M_1 commutes with M_0, it acts on S from the left:

$$s^0k \mapsto m_1s^0k = s^0m_1k, \tag{1.13}$$

in fact, it is the multiplication of a vector s by a number $\exp(iu)$ if $m_1 = \exp uY_0$.

Introduce on S the coordinate α:

$$s_1 = e^{i\alpha}.$$

Let us take on S the Euclidean measure ds:

$$ds = d\alpha\, dv, \tag{1.14}$$

where dv is the Euclidean measure on the unit sphere S^{2q-1}, so that the volume of the whole S is equal

$$\mathrm{vol}\ S = 2\pi\ \Omega_{2q} = \frac{4\pi^n}{\Gamma(n-1)},$$

recall that the volume of the unit sphere in $\mathbb{R}^m$ is equal to

$$\Omega_m = \frac{2\pi^{m/2}}{\Gamma(m/2)}. \tag{1.15}$$

Under the action (1.9) the measure ds is transformed as follows:

$$d\widetilde{s} = |(sg)_1|^{-2q}\, ds. \tag{1.16}$$

In particular, this measure is invariant with respect to K. In section 5 we shall need the following formula of an "integration by parts":

$$\int_S \psi(s)\,(L_X\varphi)(s)\,ds = 2q \int_S \mathrm{Re}\,\langle sX, s\rangle\,\psi(s)\,\varphi(s)\,ds - \int_S (L_X\psi)(s)\cdot\varphi(s)ds \tag{1.17}$$

for functions φ, ψ invariant under the left action of M_1.

Let $\widetilde{G}$ denote the universal covering group for the group G. Let $\pi : \widetilde{G} \to G$ be the natural projection. For a subgroup of G we denote by the same letter with widetilde a corresponding analytic subgroup of $\widetilde{G}$ (i.e. having the same Lie algebra). In virtue of their simple connectedness the subgroups $\widetilde{K}^{(s)}, \widetilde{A}, \widetilde{Z}, \widetilde{M_0}$ of $\widetilde{G}$ are isomorphic to the subgroups $K^{(s)}, A, Z, M_0$ of G and can be identified with them respectively. We shall assume that.

The centre $Z(\widetilde{G})$ of the group $\widetilde{G}$ is isomorphic to $\mathbb{Z}$ and is $\exp 2\pi\mathbb{Z}X_0$, see (1.5). The kernel of the projection π is the subgroup $D = \exp 2\pi n\mathbb{Z}X_0$ of $\widetilde{G}$.

Lemma 1.1 *The centre $Z(\widetilde{G})$ contains in the subgroup $\widetilde{M}$ of $\widetilde{G}$:*

$$Z(\widetilde{G}) \subset \widetilde{M}.$$

Proof: Let X be an arbitrary element of $\mathfrak{m}$, see (1.8). Decompose it according to (1.6):

$$X = \alpha X_0 + Y, \tag{1.18}$$

where $Y \in \mathfrak{k}^{(s)}$. We have $\alpha = -(n/q)u$ and

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & iv + (iu/q)E & 0 \\ 0 & 0 & iun/q \end{pmatrix}.$$

Let us take u so that $\alpha = 2\pi l,\ \ l \in \mathbb{Z}$, i.e. $u = -2\pi lq/n$, then

$$Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & iv - (2\pi il/n)E & 0 \\ 0 & 0 & -2\pi il \end{pmatrix}.$$

For this element, we have in G:

$$\exp Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp{(iv)}\cdot\exp{(-2\pi il/n)} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that $\exp Y \in M_0$. But $M_0 = \widetilde{M_0}$, hence $\exp Y \in \widetilde{M_0}$. Apply to (1.18) the map $\exp : \mathfrak{g} \to \widetilde{G}$. We obtain

$$\exp \alpha X_0 = \exp X \cdot \exp{(-Y)}. \tag{1.19}$$

If we take α as above, then both factors at the right hand side of (1.19) belong to $\widetilde{M}$, so the left hand side belongs to $\widetilde{M}$ too. □

Let us denote $\widetilde{S} = \widetilde{K}/\widetilde{M_0} = \widetilde{K}/M_0$. It follows from the lemma that

$$\widetilde{K}/\widetilde{M} = K/M. \tag{1.20}$$

The Iwasawa type decompositions of G and $\widetilde{G}$ are:

$$G = AZK, \quad \widetilde{G} = \widetilde{A}\widetilde{Z}\widetilde{K} = AZ\widetilde{K},$$

i.e. any element $g \in G$ and any element $\widetilde{g} \in \widetilde{G}$ can be written as

$$g = azk = \exp tA_0 \cdot zk \tag{1.21}$$
$$\widetilde{g} = az\widetilde{k} = \exp tA_0 \cdot z\widetilde{k} \tag{1.22}$$

In each of these decompositions the elements $a \in A$ are determined by g and $\widetilde{g}$ uniquely. Moreover, if $g = \pi(\widetilde{g})$, then the parameter t for both decompositions (1.21), (1.22) is the same. This parameter can be found as follows. Apply (1.21) to s^0:

$$s^0 g = e^t \cdot s^0 k$$

whence using $|s_1| = 1$ we get

$$e^t = |(s^0 g)_1|$$

and, therefore, for t from (1.22) we have

$$e^t = |(s^0 \pi(\widetilde{g}))_1| \tag{1.23}$$

2 Representations associated with the cone

Let $\sigma, \tau \in \mathbb{C}$. Define the following characters (one-dimensional representations) of the subgroups $A, \widetilde{K}, \widetilde{MAZ}$:

$$\begin{aligned} \omega_\sigma(a) &= e^{\sigma t}, \\ \omega_\tau(\widetilde{k}) &= e^{i\tau u}, \\ \omega_{\sigma,\tau}(\widetilde{m}az) &= \omega_\tau(\widetilde{m})\, \omega_\sigma(a) \\ &= e^{i\tau u} e^{\sigma t}, \end{aligned}$$

where $a = \exp tA_0 \in A$, $\widetilde{k} = \exp X \in \widetilde{K}$ with X given by (1.4), $\widetilde{m} = \exp X \in \widetilde{M}$ with X given by (1.8) (recall $\widetilde{M} \subset \widetilde{K}$). The subgroup $\widetilde{MAZ}$ of $\widetilde{G}$ is a maximal parabolic subgroup.

Let $T_{\sigma,\tau}$ denote the representation of $\widetilde{G}$ induced by the character $\omega_{\sigma,\tau}$ of $\widetilde{MAZ}$. It acts on the space $\mathcal{D}_{\sigma,\tau}(\widetilde{G})$ of functions φ in $C^\infty(\widetilde{G})$ satisfying the condition

$$\varphi(\widetilde{m}az\widetilde{g}) = \omega_{\sigma,\tau}(\widetilde{m}az)\, \varphi(\widetilde{g})$$

by right translations

$$T_{\sigma,\tau}(\widetilde{g})\, \varphi(\widetilde{g}_1) = \varphi(\widetilde{g}_1\widetilde{g})$$

Let us show the realization of $T_{\sigma,\tau}$ on functions on S. For a function $\varphi \in \mathcal{D}_{\sigma,\tau}(\widetilde{G})$, let us consider the following function ψ on $\widetilde{K}$:

$$\psi(\widetilde{k}) = \varphi(\widetilde{k})\,\omega_\tau(\widetilde{k})^{-1}.$$

It satisfies the condition

$$\psi(\widetilde{m}\widetilde{k}) = \psi(\widetilde{k})$$

so, by (1.20), defines a function on K/M. But it is more convenient instead of this function on K/M to consider the corresponding function on $S = K/M_0$ which is invariant with respect to the left action of M_1, see (1.13). Namely, let us consider the following function f on S defined by

$$f(s) = f(s^0 k) = \psi(\widetilde{k}) = \varphi(\widetilde{k})\,\omega_\tau(\widetilde{k})^{-1},$$

where $k = \pi(\widetilde{k})$. It satisfies the condition

$$f(\lambda s) = f(s), \quad \lambda \in \mathbb{C}, \quad |\lambda| = 1. \tag{2.1}$$

The space of functions f in $\mathcal{D}(S)$ (for a manifold M, $\mathcal{D}(M)$ denote the Schwartz space of compactly supported C^∞-functions with a usual topology) satisfying (2.1) will be denoted by V. Thus, $\mathcal{D}_{\sigma,\tau}(\widetilde{G})$ is isomorphic to V. In this realization the representation $T_{\sigma,\tau}$ looks as follows:

$$(T_{\sigma,\tau}(\widetilde{g})f)(s) = f\left(\frac{sg}{(sg)_1}\right)\,|(sg)_1|^\sigma\,\frac{\omega_\tau(\widetilde{k}_1)}{\omega_\tau(\widetilde{k})}, \tag{2.2}$$

where k is an element in K such that $s = s^0 k$, further, $\widetilde{k}$ is an element in $\widetilde{K}$ such that $\pi(\widetilde{k}) = k$, and at last $g = \pi(\widetilde{g})$; the element $\widetilde{k}_1$ is determined by the Iwasawa decomposition

$$\widetilde{k}\widetilde{g} = \widetilde{a}_1\widetilde{z}_1\widetilde{k}_1,$$

we used (1.23). It can be checked that the right hand side of (2.2) is well-defined, i.e. it does not depend on the choice of k and $\widetilde{k}$ for given s.

It is much more convenient to deal with the representation $T_{\sigma,\tau}$ of the Lie algebra $\mathfrak{g}$ corresponding to $T_{\sigma,\tau}$ of the group $\widetilde{G}$ (we use the same symbol):

$$(T_{\sigma,\tau}(X)f)(s) = (L_X f)(s) + \frac{1}{2}\Big\{(\sigma+\tau)\langle sX, s\rangle + (\sigma-\tau)\langle s, sX\rangle\Big\}\,f(s), \tag{2.3}$$

where $X \in \mathfrak{g}$, see (1.2), L_X is the operator (1.10), the form $\langle\,.\,,\,.\,\rangle$ is given by (1.12).

The Hermitian form

$$(f_1, f_2) = \int_S f_1(s)\overline{f_2(s)}\,ds, \tag{2.4}$$

where ds is the measure (1.14) is invariant with respect to the pair $T_{\sigma,\tau}, T_{\overline{\sigma}^*,\overline{\tau}}$:

$$(T_{\sigma,\tau}(X)f_1, f_2) = -(f_1, T_{\overline{\sigma}^*,\overline{\tau}}(X)f_2); \tag{2.5}$$

here and further

$$\sigma^* = 2 - 2n - \sigma = -2q - \sigma \tag{2.6}$$

(formula (2.5) is proved by means of (1.11), (1.16)).

3 Decomposition of the space V

In this section we decompose the space V into irreducible subspaces with respect to the action of the group $K = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(q)) \cong \mathrm{U}(q)$ by translations:

$$R(k)\varphi(s) = \varphi(sk). \tag{3.1}$$

We follow [5].

First consider the unit sphere $S_2 = S^{2q-1}$ consisting of points $(s_2, \dots, s_n) \in \mathbb{C}^q$, satisfying the condition $s_2\overline{s}_2 + \dots + s_n\overline{s}_n = 1$, see section 1. As it is known [9], the space $\mathcal{D}(S_2)$ decomposes into the direct sum of the subspaces $\mathcal{H}(m)$, $m \in \mathbb{N} = \{0, 1, 2, \dots\}$. The subspace $\mathcal{H}(m)$ consists of the restrictions to S_2 of homogeneous harmonic polynomials in $x_2, \overline{x}_2, \dots, x_n, \overline{x}_n$ of degree m with complex coefficients. Denote by the same symbol $\mathcal{H}(m)$ the space of those polynomials. Let $\mathcal{P}(r), \mathcal{A}(r), \overline{\mathcal{A}}(r)$ denote the spaces of all, analytic and antianalytic homogeneous polynomials of degree r respectively. Under the natural action of $U(q)$, the spaces $\mathcal{A}(r)$ and $\overline{\mathcal{A}}(r)$ are irreducible, and the space $\mathcal{H}(m)$ decomposes into the subspaces

$$\mathcal{H}(m, v) = \left[\mathcal{A}(m - v) \otimes \overline{\mathcal{A}}(v)\right] \cap \mathcal{H}(m),$$

$v = 0, 1, \dots, m$. Let us denote by $D(r), \overline{D}(r), D(m, v)$ the corresponding representations of $U(q)$ on $\mathcal{A}(r), \overline{\mathcal{A}}(r), \mathcal{H}(m, v)$ respectively.

Lemma 3.1 *The representations $D(m, v)$ of the group* $\mathrm{U}(q)$ *are irreducible and pairwise non-equivalent. The dimension of $\mathcal{H}(m, v)$ is equal to*

$$(m + q - 1)\, \frac{\Gamma(m - v + q - 1)\, \Gamma(m + q - 1)}{\Gamma(m - v + 1)\, \Gamma(q - 1)\, \Gamma(q)}$$

Proof. The highest weights of $D(r)$ and $\overline{D}(r)$ are $(r, 0, \dots, 0), (0, 0, \dots, -r)$, respectively. The highest component $D'(m, v)$ in $D(m - v) \otimes \overline{D}(v)$ has the highest weight

$$(m - v, 0, \dots, 0, -v). \tag{3.2}$$

As it is known [9],

$$\mathcal{P}(m) = \mathcal{H}(m) + Q \cdot \mathcal{H}(m - 2)$$

where $Q = x_2\overline{x}_2 + \dots + x_n\overline{x}_n$. Therefore,

$$\mathcal{A}(m - v) \otimes \overline{\mathcal{A}}(v) = \mathcal{H}(m, v) + Q \cdot \left[\mathcal{A}(m - v - 1) \otimes \overline{\mathcal{A}}(v - 1)\right]. \tag{3.3}$$

The highest weight of the second term in the right hand side is $(m - v - 1,\ 0, \dots, 0, -v + 1)$, it is lower than (3.2). Therefore, $D'(m, v) \subset D(m, v)$. But the dimensions of both latter representations are equal to each other: one can check this statement calculating these dimensions by means of (3.3) and the Weyl formula, respectively. Thus, $D(m, v) = D'(m, v)$ whence $D(m, v)$ is irreducible. The non-equivalence follows from (3.2). □

Lemma 3.2 *The restriction of the representations $D(m, v)$ to the group* $\mathrm{SU}(q)$ *are irreducible. For $q > 2$, they are pairwise non-equivalent; for $q = 2$, the restrictions with the same m are equivalent (to the representation with the highest weight $(m, 0)$).*

Indeed, the restriction of $D(m,v)$ to the group $\mathrm{SU}(q)$ has the highest weight $(m, v, \dots, v, 0)$.
□

Lemma 3.3 *The space of functions in $\mathcal{H}(m,v)$ depending only on s_n is one-dimensional. The normalized by* 1 *at the point* $(0,\dots,0,1)$ *basis function is $\psi(m,v;s_n)$, where for $m \geqslant 2v$:*

$$\psi(m,v;t) = c(m,v)\ t^{m-2v}\ F(-v, m-v+q-1; m-2v+1; t\overline{t})$$

and for $m \leqslant 2v$:

$$\psi(m,v;t) = \psi(m, m-v; \overline{t}),$$

here F is the Gauss hypergeometric function ([2], Ch. 2),

$$c(m,v) = (-1)^v\ \frac{\Gamma(m-v+1)\,\Gamma(q-1)}{\Gamma(m-2v+1)\,\Gamma(m+q-1)}.$$

Proof. Let $m \geqslant 2v$. Then a polynomial $\varphi(x)$ in $\mathcal{H}(m,v)$ depending only on x_n can be written as $x_n^{m-2v}\ f(a)$, where $f(a)$ is a polynomial in $a = x_n\overline{x}_n$ of degree v. Since φ is harmonic, f has to satisfy the equation (hypergeometric):

$$a(1-a)\frac{d^2 f}{da^2} + \big[m-2v+1-(m-2v+q)a\big]\frac{df}{da} + v(m-v+q-1)f = 0$$

and be regular at $a = 0$. □

As to the circle $S_1 = S^1$, see Section 1, the space $\mathcal{D}(S_1)$ decomposes into the sum of one-dimensional spaces $\mathcal{H}(l)$, $l \in \mathbb{Z}$, spanned by s_1^l.

Going to $S = S_1 \times S_2$, we have got that the space $\mathcal{D}(S)$ decomposes into the sum of the spaces $\mathcal{H}(l) \otimes \mathcal{H}(m,v)$. They are irreducible under $K \cong U(q)$ and even $K^{(s)} \cong \mathrm{SU}(q)$.

Now let us decompose V (see Section 2). The condition (2.1) gives

$$l + m = 2v, \tag{3.4}$$

whence

$$l \equiv m\ \ (mod\ 2) \tag{3.5}$$

and

$$|l| \leqslant m. \tag{3.6}$$

Let us denote by Λ the set of pairs $z = (l,m)$, $l, m \in \mathbb{Z}$, which satisfy (3.5) and (3.6). These pairs will be called *weights*. Thus, K-irreducible subspaces which occur in the decomposition of V can be labelled by weights $z \in \Lambda$. We shall write $\mathcal{H}(z)$ for $\mathcal{H}(l) \otimes \mathcal{H}(m,v)$ with (3.4). The corresponding representation of K and $\mathfrak{k}$ on $\mathcal{H}(z)$ will be denoted by $R(z)$.

For each $\mathcal{H}(z)$, the subspace of functions in $\mathcal{H}(z)$ depending only on s_1 and s_n is one-dimensional, the basis function $\psi_z(s_1, s_n)$ normalized by 1 at s^0 is for $l \geqslant 0$:

$$\psi_z(s_1, s_n) = s_1^l\ c(l,m)\ \overline{s}_n^l\ F\left(\frac{l-m}{2}, \frac{l+m+2q-2}{2}; l+1; s_1\overline{s}_n\right) \tag{3.7}$$

and for $l \leqslant 0$:

$$\psi_{(l;m)}(s_1, s_n) = \overline{\psi_{(-l;m)}(s_1, s_n)} \tag{3.8}$$

where for $l \geqslant 0$:

$$c(l,m) = (-1)^{(m-l)/2} \frac{\Gamma((l+m+2)/2)\,\Gamma(q-1)}{\Gamma(l+1)\,\Gamma((-l+m+2q-2)/2)}. \tag{3.9}$$

4 The structure of representations associated with the cone

For the study of the structure of the representations $T_{\sigma,\tau}$ (irreducibility, composition series etc.) we use the restriction to $\mathfrak{k}$. Let $X \in \mathfrak{k}$, see (1.4). Then by (2.3) we have

$$T_{\sigma,\tau}(X) = L_X + i\tau u. \tag{4.1}$$

Remember the decomposition (1.6) and consider (4.1) when X belongs to one of two terms of (1.6).

For $X = X_0$, see (1.5), we have

$$T_{\sigma,\tau}(X_0) = L_{X_0} + \frac{i\tau q}{n}.$$

so that

$$T_{\sigma,\tau}(X_0) = -\frac{i}{n}(q\tau + nl) \quad \text{on} \quad \mathcal{H}(z). \tag{4.2}$$

For $X \in \mathfrak{k}^{(s)}$, we have $T_{\sigma,\tau}(X) = L_X$, so that the restriction of $T_{\sigma,\tau}$ of $\widetilde{G}$ to the subgroup $K^{(s)} \cong \widetilde{K}^{(s)} \cong SU(q)$ does not depend on σ, τ and is the representation R of SU(q) on V by translations, see (3.1).

Lemma 4.1 *The restriction of the representation $T_{\sigma,\tau}$ to the Lie subalgebra $\mathfrak{k}$ decomposes into the direct sum of irreducible pairwise non-equivalent representations $R(z)$ on the spaces $\mathcal{H}(z)$, see Section 3.*

The lemma follows from Lemma 3.2 and (4.2).

Since A_0 commutes with $\mathfrak{m}$, see (1.7) and (1.8), the operator $T_{\sigma,\tau}(A_0)$ preserves the space of functions depending on $s_1 = x, s_n = y$ only. On such functions this operator has the following expression:

$$y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \overline{y}\frac{\partial}{\partial \overline{x}} + \overline{x}\frac{\partial}{\partial \overline{y}} - \operatorname{Re}(x\overline{y})\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \overline{x}\frac{\partial}{\partial \overline{x}} + \overline{y}\frac{\partial}{\partial \overline{y}}\right) +$$

$$+\frac{1}{2}\{(\sigma+\tau)\overline{x}y + (\sigma-\tau)x\overline{y}\}.$$

Apply it to ψ_z, see (3.7), (3.8), (3.9). Omitting rather cumbersome calculations with hypergeometric functions, we write down the result.

Introduce the following 4 vectors on the plane $\mathbb{R}^2$:

$$e_1 = (1,1), \quad e_2 = (1,-1), \quad e_3 = (-1,1), \quad e_4 = (-1,-1)$$

and the following 4 linear functions β_i of z:

$$\beta_1(\sigma,\tau;z) = \sigma-\tau-l-m, \quad (4.3)$$
$$\beta_2(\sigma,\tau;z) = \sigma-\tau-l+m+2q-2, \quad (4.4)$$
$$\beta_3(\sigma,\tau;z) = \sigma+\tau+l+m, \quad (4.5)$$
$$\beta_4(\sigma,\tau;z) = \sigma+\tau+l+m+2q-2. \quad (4.6)$$

For $i=1,2,3,4$ we shall denote $i'=5-i$, so that $e_{i'}=-e_i$. Notice that

$$\beta_i(\sigma,\tau;z)+\beta_{i'}(\sigma^*,\tau;z)=-2.$$

Theorem 4.2 *We have*

$$T_{\sigma,\tau}(A_0)\,\psi_z=\sum_{i=1}^{4}\gamma_i(z)\beta_i(\sigma,\tau;z)\psi_{z+e_i} \quad (4.7)$$

where

$$\gamma_i(z)=(-1)^{i'}\,\frac{\beta_{i'}(0,0;z)}{4(m+q-1)}.$$

Remark that $\gamma_1(z)$ and $\gamma_3(z)$ have no zeroes on Λ (for Λ, see Section 3) and $\gamma_2(z)$ and $\gamma_4(z)$ vanish on the rays $l=m$ and $m=-l$ in Λ, respectively.

We say that the representation $T_{\sigma,\tau}$ links a weight z with a weight z' if there exist $X\in\mathfrak{g}$ and $f\in\mathcal{H}(z)$ such that $T_{\sigma,\tau}(X)f$ is non-zero and belongs to $\mathcal{H}(z')$.

Lemma 4.3 *The representation $T_{\sigma,\tau}$ links z and z' if and only if $z'=z+e_i$ with $\beta_i(\sigma,\tau;z)\neq 0$ for some i.*

The lemma is proved similarly to the corresponding lemma from [5].

Let us call the line $\beta_i(\sigma,\tau;z)=0$ on the plane of $z=(l,m)$ *a barrier* for $T_{\sigma,\tau}$ if it intersects with Λ for $i=1,3$ and with $\Lambda\cap\{m>|l|\}$ for $i=2,4$. The barrier $\beta_i=0$ divides Λ into two parts: the set $\beta_i\geqslant 0$ (the interior of the barrier) and the set $\beta_i<0$ (the exterior of the barrier). If $\beta_i=0$ is a barrier, we shall sometimes speak for brevity: there is the barrier i. Let us indicate sets of pairs (σ,τ) for which there is the barrier i (see Fig. 1):

barrier 1: $\sigma-\tau=0,2,4,\ldots$

barrier 2: $\sigma-\tau=-2q,-2q-2,\ldots$

barrier 3: $\sigma+\tau=0,2,4,\ldots$

barrier 4: $\sigma+\tau=-2q,-2q-2,\ldots$

So we see that for given σ,τ at most 2 barriers may happen.

If $\beta_i=0$ is a barrier then we denote by V_i the subspace of V which is the direct sum of $\mathcal{H}(z)$ with $\beta_i\geqslant 0$.

The following theorem is a quick implication of Lemma 4.3.

Theorem 4.4 *The subspaces V_i are invariant under $T_{\sigma,\tau}$. Any irreducible subfactor of $T_{\sigma,\tau}$ is obtained by means of subspaces V_i (i.e. any irreducible subfactor is V, or V_i, or $V_i\cap V_j$ factorized over a sum of several V_k). If both $\sigma+\tau$ and $\sigma-\tau$ do not belong to $2\mathbb{Z}$, then $T_{\sigma,\tau}$ is irreducible.*

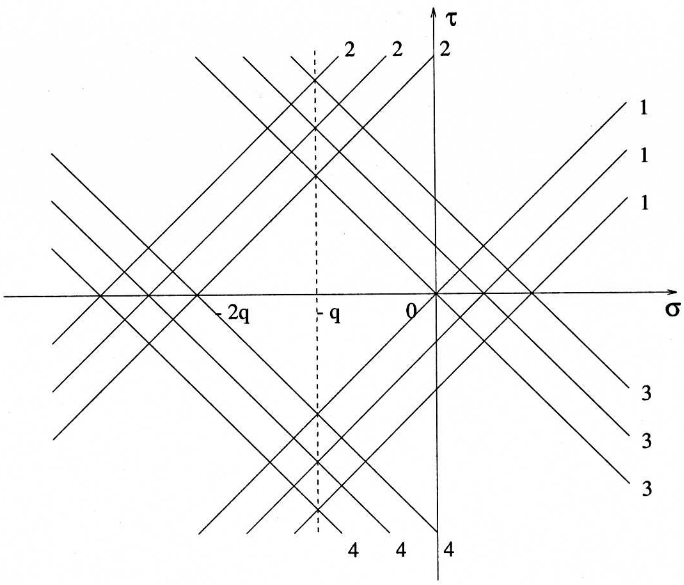

Fig. 1

In order to see the structure of invariant subspaces, irreducible subfactors, composition series etc., it is sufficient to draw barriers on the plane of $z = (l, m)$ and to endow each barrier by a bristle (dashes) oriented inside of the barrier (where $\beta_i \geqslant 0$), see Fig. 2–5 for the cases with two barriers (Fig. 4, 5 are given for $\sigma > 1 - q$).

Summarizing, we can write that when there are 2 barriers i, j the irreducible subfactors are:

$$V_i \cap V_{j'}, \quad V_i/V_{j'}, \quad V_{j'}/V_i, \quad V/(V_i + V_{j'}), \tag{4.8}$$

for brevity we write V_i/V_j instead of $V_i/(V_i \cap V_j)$, the sum in (4.8) means the arithmetic sum. In fact, some subfactors indicated in (4.8) can be absent (be trivial). It happens for the cases $11'$ and $22'$: then either the first, or the last one in (4.8) is trivial, and if, moreover, $\sigma = -q$, i.e.

$$\sigma = -q, \quad \tau = q, q+2, \ldots, \tag{4.9}$$

or

$$\sigma = -q, \quad \tau = -q, -q-2, \ldots, \tag{4.10}$$

then both are trivial, so that V decomposes into the direct sum

$$V = V_1 + V_{1'} \text{ or } V = V_2 + V_{2'} \tag{4.11}$$

respectively to (4.9) or (4.10).

The irreducible subspace $V_i \cap V_{i'}$ for $\sigma = 1 - q$ has weights z lying on one ray, i.e. on coinciding barriers i, i'. Similarly it goes for $\sigma = -1 - q$ for the subfactor $V/(V_i + V_{i'})$. Let us call such subfactors *exceptional*.

The subfactors $V_1 \cap V_{2'}$ and $V/(V_2 + V_{1'})$ are finite-dimensional. In particular, for $\sigma = \tau = 0$ and $\sigma^* = \tau = 0$ they realize the unit representation of $\widetilde{G}$ (and G).

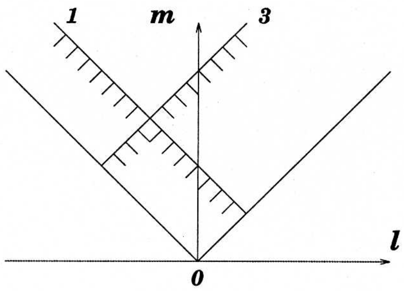

Fig. 2

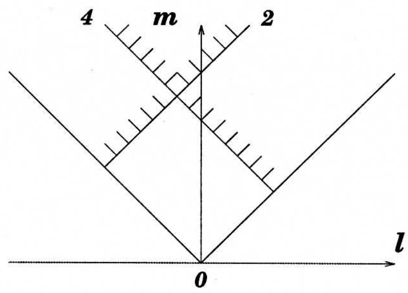

Fig. 3

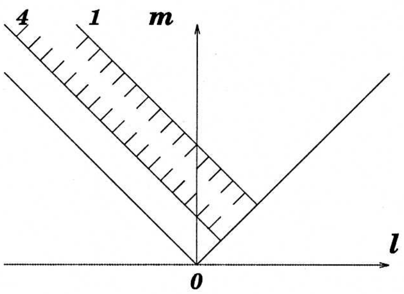

Fig. 4

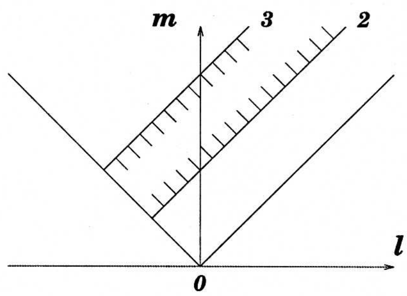

Fig. 5

For the subfactor W, let $\Lambda(W)$ denote the set of weights $z \in \Lambda$ occuring in W. For an invariant subspace U under $T_{\sigma,\tau}$, denote by $U^\perp$ its orthogonal complement with respect to form (2.4). Then $\Lambda(U^\perp) = \Lambda \setminus \Lambda(U)$ and $U^\perp$ is invariant under $T_{\overline{\sigma}^*,\overline{\tau}}$. For a subfactor $W = U/Z$ of $T_{\sigma,\tau}$ the subspace $W^* = Z^\perp/U^\perp$ is a subfactor of $T_{\overline{\sigma}^*,\overline{\tau}}$. We have $\Lambda(W^*) = \Lambda(W)$. Let us call W^* the *dual* subfactor. The dual subfactors for V_i and V/V_i are $V/V_{i'}$ and $V_{i'}$ respectively, and for the subfactors (4.8) are $V/(V_{i'} + V_j)$, $V_j/V_{i'}$, $V_{i'}/V_j$, $V_{i'} \cap V_j$ respectively.

5 Intertwining operators

A continuous operator A on V is said to intertwine representations $T_{\sigma,\tau}$ and T_{σ_1,τ_1} if for any $X \in \mathfrak{g}$ we have

$$T_{\sigma_1,\tau_1}(X)\, A = A\, T_{\sigma,\tau}(X). \tag{5.1}$$

A similar definition is given for the case when an operator A maps a subfactor W of $T_{\sigma,\tau}$ in a subfactor W_1 of T_{σ_1,τ_1}.

In this section we find out all such operators and show "the main of them" in an integral form.

Theorem 5.1 *A non-zero intertwining operator A as above exists only in the following cases:*

$$(a)\quad \sigma_1 = \sigma, \quad \tau_1 = \tau;$$
$$(b)\quad \sigma_1 = \sigma^*, \quad \tau_1 = \tau.$$

In the case (a) *such an operator is a scalar operator (the multiplication by a number) on V except of* (4.9), (4.10), *then A is a scalar operator on each term in* (4.11).

In the case (b) *for any irreducible subfactor W of $T_{\sigma,\tau}$ there exists a unique up to the factor non-zero operator mapping W onto the dual subfactor W^* and intertwining the subfactor of $T_{\sigma,\tau}$ on W with the subfactor of $T_{\sigma^*,\tau}$ on W^*.*

Proof. Let A be a continuous operator on V satisfying (5.1). It follows from Lemma 4.1, that A preserves each $\mathcal{H}(z)$ and its restriction to $\mathcal{H}(z)$ is the multiplication by a number $a(z)$. In particular, we have

$$A\psi_z = a(z)\psi_z. \tag{5.2}$$

The numbers $a(z)$ depend on $\sigma, \tau, \sigma_1, \tau_1$. Take in (5.1) $X = X_0$, see (1.5), and use (4.2), we just obtain that $\tau_1 = \tau$. Now take in (5.1) $X = A_0$, see (1.7), apply to ψ_z and use (4.7), then we obtain relations for $a(z)$:

$$\beta_i(\sigma, \tau; z)\, a(z + e_i) = \beta_i(\sigma_1, \tau; z)\, a(z), \tag{5.3}$$

$i = 1, 2, 3, 4$. These relations have to be co-ordinated, i.e. if one applies to $a(z + e_i)$ the same relation with i replaced by i', then one has to get $a(z)$ again. It gives the conditions:

$$\beta_i(\sigma, \tau; z + e_{i'})\, \beta_{i'}(\sigma, \tau; z)\, a(z) = \beta_i(\sigma_1, \tau; z + e_{i'})\, \beta_{i'}(\sigma_1, \tau; z)\, a(z). \tag{5.4}$$

Since $A \neq 0$, there exists at least one z such that $a(z) \neq 0$. Omitting this $a(z)$ from (5.4) and substituting the explicit expressions (4.3) – (4.6), we obtain that a square trinomial (in σ) whose roots give the sum $-2q$ has the same values at σ and σ_1. Therefore, we have $\sigma_1 = \sigma$ or $\sigma_1 = -\sigma - 2q \ (= \sigma^*)$.

First let $\sigma_1 = \sigma$. The relations (5.3) shows that $a(z+e_i) = a(z)$ for that z and i for which $\beta_i(\sigma, \tau; z) \neq 0$. This fact together with the same one for i' gives that $a(z)$ should be constant along any line $z + te_i$ and, therefore, on the whole Λ, except the case (4.9), (4.10) when two barriers $11'$ or $22'$ split Λ into two non-linked subsets, so that we have decompositions (4.11).

Now let $\sigma_1 = \sigma^*$. Then (5.3) is

$$\beta_i(\sigma, \tau; z)\, a(z + e_i) = \beta_i(\sigma^*, \tau; z)\, a(z). \tag{5.5}$$

Let $W = U/Z$ be some irreducible subfactor of $T_{\sigma,\tau}$. Similarly to [4] we can show that equations (5.5) have a unique up to the factor solution on $\Lambda(U)$ which is the intersection of the interiors of some barriers and, moreover, $a(z) = 0$ on $\Lambda(U) \cap \Lambda(Z)$. This solution $a(z)$ defines an operator on U vanishing on $U \cap Z$ so that it maps U/Z onto $Z^\perp/U^\perp$ and intertwines corresponding subfactors of $T_{\sigma,\tau}$ and $T_{\sigma^*,\tau}$. Similarly to [4] it is shown that this operator is continuous. □

Denote by $z^{\lambda,\tau}$ where $z, \lambda \in \mathbb{C}$, $\tau \in \mathbb{R}$, the following function of z: if $z = re^{i\alpha}$, $r > 0$, $0 \leqslant \alpha < 2\pi$, then

$$z^{\lambda,\tau} = r^\lambda e^{i\tau\alpha}.$$

If $z = z(u)$ is some curve, then

$$\left.\frac{d}{du}\right|_{u=0} z^{\lambda,\tau} = \left\{ z^{\lambda,\tau} \left(\lambda \ \mathrm{Re}\, \frac{z'}{z} + i\tau \ \mathrm{Im}\, \frac{z'}{z} \right) \right\}\bigg|_{u=0}. \tag{5.6}$$

Now let us consider the operator $A_{\sigma,\tau}$ on V which is defined by integral:

$$(A_{\sigma,\tau} f)(s) = \int_S \left[\frac{s}{s_1}, \frac{t}{t_1} \right]^{\sigma^*,\tau} f(t)\, dt. \tag{5.7}$$

The integral converges absolutely for $\mathrm{Re}\,\sigma < -n + 1$ and is extended to other σ, τ by the analyticity.

Theorem 5.2 *The operator $A_{\sigma,\tau}$ intertwines $T_{\sigma,\tau}$ and $T_{\sigma^*,\tau}$. On the subspaces $\mathcal{H}(z)$ it is the multiplication by the numbers*

$$a(\sigma,\tau;z) = 4\,(-1)^m\,\pi^n\,e^{i\tau\pi}\,\frac{\Gamma(1-n-\sigma)\Gamma(2-n-(\sigma+\tau)/2)\Gamma(2-n-(\sigma-\tau)/2)}{\prod\limits_{i=1}^{4}\Gamma(-(1/2)\beta_i(\sigma,\tau;z))}. \tag{5.8}$$

Proof. First we show that $A_{\sigma,\tau}$ is an intertwining operator. Denote by $A(s,t)$ the kernel (a function) of this operator.

Let $X \in \mathfrak{g}$. According to (1.9) we write $\widetilde{s} = s\cdot\exp uX$, $\widetilde{t} = t\cdot\exp uX$. Denote $a = \langle sX, s\rangle$, $b = \langle tX, t\rangle$, see (1.12). Apply (5.1) with $\sigma_1 = \sigma^*$, $\tau_1 = \tau$ to a function $f \in V$. Then the left hand side is:

$$\int \frac{d}{du}\bigg|_{u=0} A(\widetilde{s},t)f(t)\,dt + \frac{1}{2}\left[(\sigma^*+\tau)a + (\sigma^*-\tau)\overline{a}\right]\int A(s,t)\,\varphi(t)\,dt \tag{5.9}$$

and the right hand side is:

$$\int A(s,t)\left\{(L_X f)(t) + \frac{1}{2}\left[(\sigma+\tau)b + (\sigma-\tau)\overline{b}\right]\,f(t)\right\}\,dt. \tag{5.10}$$

By (1.17) the difference between (5.9) and (5.10) is equal to

$$\int\left\{\frac{d}{du}\bigg|_{u=0} A(\widetilde{s},\widetilde{t}) + A(s,t)\,\big[-2q\,\mathrm{Re}\,b + \right.$$
$$\left. +\frac{1}{2}\big((\sigma^*+\tau)a + (\sigma^*-\tau)\overline{a}\big) - \frac{1}{2}\big((\sigma+\tau)b + (\sigma-\tau)\,\overline{b}\big)\big]\right\}\,f(t)\,dt \tag{5.11}$$

since

$$\frac{d}{du}\bigg|_{u=0}\big(A(\widetilde{s},t) + A(s,\widetilde{t})\big) = \frac{d}{du}\bigg|_{u=0} A(\widetilde{s},\widetilde{t}).$$

By (5.6) we have

$$\begin{aligned}\frac{d}{du}\bigg|_{u=0} A(\widetilde{s},\widetilde{t}) &= \frac{d}{du}\bigg|_{u=0}\left\{\frac{[s,t]}{(sg)_1\overline{(tg)}_1}\right\}^{\sigma^*,\tau}\\ &= A(s,t)\left\{-\sigma^*\,\mathrm{Re}\,(a+\overline{b}) - i\tau\,\mathrm{Im}\,(a+\overline{b})\right\}.\end{aligned}$$

Substituting it in (5.11), we obtain that the integrand of (5.11) is $A(s,t)f(t)$ multiplied by

$$-\sigma^*\,\mathrm{Re}\,(a+\overline{b}) - i\tau\,\mathrm{Im}\,(a+\overline{b}) - 2q\mathrm{Re}b + \frac{1}{2}\left[(\sigma^*+\tau)a + (\sigma^*-\tau)\overline{a}\right] - \frac{1}{2}\left[(\sigma+\tau)b + (\sigma-\tau)\overline{b}\right]$$

which is equal to 0 in virtue of (2.6). It proves that our operator is intertwining.

Therefore, as it was said in the proof of Theorem 5.1, the operator $A_{\sigma,\tau}$ on the subspace $\mathcal{H}(z)$ is the multiplication by a number $a(\sigma,\tau;z)$, or $a(z)$ for brevity.

First we calculate $a(0)$. Set in (5.2) $A = A_{\sigma,\tau}$, $z = 0$ (so that $\psi_z = 1$) and take it at $s = s^0$. Then

$$a(0) = \int\limits_S \left(-1 + \overline{t}_n/\overline{t}_1\right)^{\sigma^*,\tau}\,dt.$$

Taking $t_1 = e^{i\alpha}$, $t_n = re^{i\beta}$ and integrating in other variables, we obtain

$$a(0) = \Omega_{2q-2} \int_0^{2\pi}\int_0^{2\pi}\int_0^1 \left(-1 + e^{i\alpha} r e^{-i\beta}\right)^{\sigma^*,\tau} (1-r^2)^{q-2}\, r\, dr\, d\alpha\, d\beta$$

$$= 2\pi\Omega_{2q-2} \int_0^{2\pi}\int_0^1 \left(-1 + re^{iv}\right)^{\sigma^*,\tau} (1-r^2)^{q-2}\, r\, dr\, dv.$$

The latter integral is calculated by means of the change

$$re^{iv} - 1 = \rho e^{i\gamma}$$

(so that $\frac{\pi}{2} < \gamma < \frac{3\pi}{2}$) and formula [2] 1.5 (30). Finally we obtain

$$a(0) = 4\,\pi^n\, e^{i\tau\pi} \,\frac{\Gamma(1-n-\sigma)}{\Gamma((-\sigma-\tau)/2)\,\Gamma((-\sigma+\tau)/2)}.$$

Now, starting from $a(0)$ and applying the formula

$$a(z+e_i) = a(z)\,\frac{\beta_i\,(\sigma^*,\tau;z)}{\beta_i\,(\sigma,\tau;z)}$$

(which is (5.3) with $\sigma_1 = \sigma^*$) first $(l+m)/2$ times with $i = 1$ and then $(m - l/2$ times with $i = 3$, we obtain (5.8). □

By means of operator (5.7) we can write down for each irreducible subfactor $W = U/Z$ an intertwining operator on the dual subfactor W^*, see Theorem 5.1. Namely, it is the first non-zero Laurent, or Taylor, coefficient at $\lambda = \sigma$ of the operator $A_{\lambda,\tau}$ on U considered as a function of λ.

6 Unitarity

A sesqui-linear form $H(f, f_1)$ on V is called invariant with respect to the pair $T_{\sigma,\tau}, T_{\sigma_1,\tau_1}$, if for any $X \in \mathfrak{g}$ and any $f, f_1 \in V$ we have

$$H(T_{\sigma,\tau}(X)f, f_1) + H(f, T_{\sigma_1,\tau_1}(X)f_1) = 0. \tag{6.1}$$

Similar we define the invariance of a form on a pair of invariant subfactors W/U and W_1/U_1; then $f \in W, f_1 \in W_1$ and $H(f, f_1) = 0$ if $f \in U$ or $f_1 \in U_1$.

Theorem 6.1 *A non-zero sesqui-linear form $H(f, f_1)$ invariant with respect to the pair $T_{\sigma,\tau}, T_{\sigma_1,\tau_1}$ on V or on a pair of invariant subfactors exists only in the following cases:*

$$(a)\quad \sigma_1 = \overline{\sigma}^*, \quad \tau_1 = \overline{\tau};$$
$$(b)\quad \sigma_1 = \overline{\sigma}, \quad \tau_1 = \overline{\tau}.$$

In the case (a) the form H coincides up to the factor with the form (2.4) *except* (4.9) *and* (4.10). *In the latter cases the form H coincides up to the factor with* (2.4) *on each of subspaces* (4.11).

In the case (b) the form H can be expressed in terms of the form (2.4) and the operator A intertwining $T_{\sigma,\tau}$ and $T_{\sigma^*,\tau}$ or their subfactors, see Theorem 5.1, namely

$$H(f, f_1) = (Af, f_1).$$

Proof. First set in (6.1) $X = X_0$ and take $f, f_1 \in \mathcal{H}(z)$, then by (4.2) we obtain $\tau_1 = \overline{\tau}$. Now let X range $\mathfrak{k}^{(s)}$. Then (6.1) gives that H is invariant with respect to the representation R of $K^{(s)} \cong SU(q)$. Therefore the subspaces $\mathcal{H}(z)$ are orthogonal with respect to the form H (for $q = 2$, one has else to remember X_0 and (4.2) and to obtain the orthogonality for different l).

Therefore, H has a "diagonal form":

$$H(f, f_1) = h(z)\,(f, f_1) \tag{6.2}$$

if both f and f_1 belong to $\mathcal{H}(z)$ and $H(f, f_1) = 0$ if f, f_1 belong to different $\mathcal{H}(z)$. Further the study of $h(z)$ goes similarly to the study of $a(z)$ in the proof of Theorem 5.1, so we omit it. □

Now determine when representations $T_{\sigma,\tau}$ or their subfactors are unitarizable. For that we have to set $\sigma_1 = \sigma$ and $\tau_1 = \tau$ in Theorem 6.1 and the form $H(f, f_1)$ has to be Hermitian ($H(f, f)$ real) and positively definite. For τ we obtain $\tau \in \mathbb{R}$. For σ we obtain two possibilities: $\sigma = \overline{\sigma}^*$ and $\sigma = \overline{\sigma}$. In the first case ($\sigma = \overline{\sigma}^*$) we have Re $\sigma = 1 - n = -q$. So we obtain a series of unitarizable representations $T_{\sigma,\tau}$ of $\widetilde{G}$ (or $\mathfrak{g}$), the invariant inner product is (2.5), so that the unitary completion acts on the space of functions f in $L^2(S)$ satisfying (2.1). Let us call this series the *continuous series*. Representations of this series are irreducible except the split case (4.9), (4.10).

Let now consider the second case: $\sigma = \overline{\sigma}$, i.e. $\sigma \in \mathbb{R}$. First consider the irreducible case.

Theorem 6.2 *An irreducible representation $T_{\sigma,\tau}$ with $\sigma, \tau \in \mathbb{R}$ is unitarizable for points on the plane (σ, τ) which fill in one "big" square $|\sigma + q| + |\tau| < q$ (with the diagonal of length $2q$) and a family of "small" squares: $|\sigma + q| + |\tau \pm (q + 2k + 1)| < 1, k \in \mathbb{N}$ (with the diagonal of length 2). The invariant inner product is $c(\sigma, \tau)$ $(A_{\sigma,\tau} f, f_1)$, where $A_{\sigma,\tau}$ is the operator from section 5, $(.,.)$ is the form (2.4) and $c(\sigma, \tau) = a(\sigma, \tau; 0)^{-1}$.*

Let us call the family of representations pointed out in this theorem the *complementary series.*

Proof. As well as $a(z)$ in section 5, the factor $h(z)$ from (6.2) satisfies the equation:

$$\beta_i(\sigma, \tau; z)\ h(z + e_j) = \beta_i(\sigma^*, \tau; z)\ h(z). \tag{6.3}$$

So we have to learn when equations (6.3) have a positive solution $h(z)$. For that it is necessary and sufficient that for each i both functions $\beta_i(\sigma, \tau; z)$ and $\beta_i(\sigma^*, \tau; z)$ considered as functions of $z = (l, m)$ are of one sign on Λ. In turn, for that it is necessary and sufficient that both intervals $(\sigma - \tau, \sigma^* - \tau)$ and $(\sigma + \tau, \sigma^* + \tau)$ on $\mathbb{R}$ contain no points of $2\mathbb{N}$. Hence we obtain the cases indicated in the theorem. □

Now let us turn to the reducible case: $\sigma \pm \tau \in 2\mathbb{Z}$. From (6.3) we see that if $h(z)$ is defined outside some barrier then $h(z) = 0$ inside of it. Therefore, we have to look for invariant positively definite Hermitian forms on irreducible subfactors. Omitting rather tiresome treatments, let us give the result.

Theorem 6.3 *Unitarizable representations on irreducible subfactors of $T_{\sigma,\tau}$ form the following series:*

(a) **long thin** *series (labelled by one parameter ranging a ray on the real line) on invariant subspaces:*
V_1 *for* $\sigma - \tau = 0,\ \sigma < 0$ *and for* $\sigma - \tau = 2, 4, 6, \ldots,\quad \sigma < -q + 1;$
V_3 *for* $\sigma + \tau = 0,\ \sigma < 0$ *and for* $\sigma + \tau = 2, 4, 6, \ldots,\quad \sigma < -q + 1;$
and, in the duality, on invariant factor-spaces:
V/V_4 *for* $\sigma^* - \tau = 0,\ \sigma^* < 0$ *and for* $\sigma^* - \tau = 2, 4, \ldots,\ \sigma^* < -q + 1;$
V/V_2 *for* $\sigma^* + \tau = 0,\ \sigma^* < 0$ *and for* $\sigma^* + \tau = 2, 4, \ldots,\ \sigma^* < -q + 1;$

(b) **short thin** *series (labelled by one parameter ranging an interval on the real line) on invariant subspaces:*
V_2 *for* $\sigma^* + \tau = 0, 2, 4, \ldots,\ -q - 1 < \sigma^* < \tau;$
V_4 *for* $\sigma^* - \tau = 0, 2, 4, \ldots,\ -q - 1 < \sigma^* < -\tau;$
and, in the duality, on invariant factor-spaces:
V/V_3 *for* $\sigma + \tau = 0, 2, 4, \ldots,\ -q - 1 < \sigma < \tau;$
V/V_1 *for* $\sigma - \tau = 0, 2, 4, \ldots,\ -q - 1 < \sigma < -\tau;$

(c) **"discrete"** *series: these representations correspond to integer even points ($\sigma, \tau \in 2\mathbb{Z}$) lying on the right and the left angles (i.e. $\sigma \geqslant |\tau|$ and $\sigma^* \geqslant |\tau|$ respectively):*
$V/(V_1 + V_3)$ *for* $\sigma \geqslant |\tau|,$
and, in the duality,
$V_2 \cap V_4$ *for* $\sigma^* \geqslant |\tau|.$

(d) **"exceptional"** *series: these representations correspond to integer even points ($\sigma, \tau \in 2\mathbb{Z}$) at the ends of the thin series (the weights z for each of these representations lie on a ray), namely, in the right angle ($\sigma \geqslant |\tau|$):*
on V_1/V_3 *for* $\sigma = \tau,$
on V_3/V_1 *for* $\sigma = -\tau,$
in the left angle ($\sigma^* \geqslant |\tau|$) (dual to the right angle):
on V_2/V_4 *for* $\sigma^* = \tau,$
on V_4/V_2 *for* $\sigma^* = -\tau,$
in the upper angle ($\tau \geqslant |\sigma + q| + q$):
on $V_2 \cap V_3$ *for* $\sigma = -q + 1,$
on $V/(V_2 + V_3)$ *for* $\sigma = -q - 1$ *(dual to the preceding case);*
in the lower angle ($\tau \leqslant -|\sigma + q| - q$):
on $V_1 \cap V_4$ *for* $\sigma = -q + 1,$
on $V/(V_1 + V_4)$ *for* $\sigma = -q - 1$ *(dual to the preceding case);*

(e) *the* **unit** *representation – in the right and the left vertex of the big square:*
on $V_1 \cap V_3$ *for* $\sigma = 0,\ \tau = 0,$
on $V/(V_2 + V_4)$ *for* $\sigma^* = 0,\ \tau = 0.$

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